Last time: position and momentum bases/operators

Hamiltonian operators / time evolution

Schrödinger's equation

Given a Hamiltonian operator $\hat{H}$

acting on $L^2(\mathbb{Z}/N) = \{\psi : \mathbb{Z}/N \to \mathbb{C}\}$

(Usually $\hat{H} = \frac{1}{2}\hat{P}^2 + V(\hat{X})$ for some potential function $V : \mathbb{Z}/N \to \mathbb{C}$)

We want to solve Schrödinger's equation

$$i \frac{d}{dt} U(t) = \hat{H}\, U(t).$$

To solve this, we gave an unfinished definition of the operator exponential. Let's finish it.

Recall that $B(V)$ denotes the space of bounded linear operators on a normed space V.

$B(V)$ is a normed space with the operator norm

$$\|L\| = \inf\{C > 0 \mid \|Lv\| \leq C\|v\| \ \forall v \in V\}$$

We showed in a problem last time that every linear operator on a finite dimensional normed space is bounded, so $B(V) = \mathrm{End}(V)$.

We can now say:

Def: Let L be a bounded operator on a normed space V.
Then the exponential of L is

$$\exp(L) = \sum_{n=0}^{\infty} \frac{L^n}{n!}$$

when this series converges in the norm topology on $B(V)$.

Today we will prove parts of the following theorem (some of which might be HW)
Recall that a normed space is Banach if every Cauchy sequence converges.
A sequence (v_n) in V is Cauchy if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n,m \geq N \ \|v_n - v_m\| < \varepsilon.$$

Thm: Let V be a Banach space. Then the exponential is defined for every $L \in B(V)$, and $\exp(L) \in B(V)$. Furthermore, it satisfies the following.

1) $\exp(L)v = \sum_{n=0}^{\infty} \frac{1}{n!}(L^n v) \quad \forall v \in V$

← series in V

2) If L_1, L_2 commute, then

$$\exp(L_1 + L_2) = \exp(L_1)\exp(L_2) = \exp(L_2)\exp(L_1)$$

if they don't commute, see Baker-Campbell-Hausdorff formula

3) The map $t \mapsto \exp(tL)$ defines a smooth function

$$\mathbb{R} \to B(V)$$

such that $\frac{d}{dt}\exp(tL) = L\exp(tL)$.

4) If v is an eigenvector of L

with eigenvalue λ, then $\exp(L)v = e^{\lambda}v$.

The upshot of this is that, assuming V is a Banach space and $\hat{H}$ is bounded, the equation

$$\frac{d}{dt}U(t) = -i\hat{H}U(t)$$

is solved by $U(t) = A\exp(-it\hat{H})$

any time-independent operator

Also, these solutions are the only ones.

Furthermore, if $e_1, \ldots, e_n$ is a basis for V which diagonalizes $\hat{H}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$ resp., then

$$U(t)v = U(t)\sum_{j=1}^{n} v_j e_j$$

$$= \sum_{j=1}^{n} v_j \exp(-it\lambda_j)\, e_j \quad .$$

If V is a quantum system and $\hat{H}$ is the Hamiltonian, this says that we can find the time evolution of a state $|\psi\rangle$ by expanding $|\psi\rangle$ in a basis of energy eigenstates $|E_j\rangle$ and we find

$$U(t)|\psi\rangle = \sum_{j=1}^{n} \langle E_j|\psi\rangle \exp(-itE_j)|E_j\rangle$$

To see why we can apply this to the "particle on $G = \mathbb{Z}/N$" system (or any finite dimensional system) we have a problem: any finite dimensional normed space is Banach

How does the state $\mathbb{1}|x=0\rangle$ evolve over time in the "particle on $\mathbb{Z}/N$ with $V=0$" system? $\hat{H} = \frac{1}{2}\hat{P}^2$

For now work with an arbitrary state $|\psi\rangle$. Need to expand $|\psi\rangle$ in the energy eigenbasis. With $\hat{H} = \frac{1}{2}\hat{P}^2$, we found that the energy eigenbasis = momentum eigenbasis $|k\rangle$

We computed $\langle k|\psi\rangle = \tilde{\psi}(k)$ previously (the DFT)

We find that $U(t)|\psi\rangle = \sum_{k=0}^{N-1} \tilde{\psi}(k)\, e^{-itE_k}\,|k\rangle$

where $E_k = \frac{1}{2}\sin^2\left(\frac{2\pi}{N}k\right)$.

If $|\psi\rangle = |x=0\rangle$, then

$$\begin{aligned}\langle k|0\rangle &= \left(\langle 0|k\rangle\right)^* \\ &= \left(\tfrac{1}{\sqrt{N}}\right)^* \\ &= \tfrac{1}{\sqrt{N}} \qquad \forall k \in \{0, \dots, N-1\}\end{aligned}$$

To compute the position coefficients (i.e. the wavefunction) at time t, we find

$$\begin{aligned}\psi_t(x) &\doteq \langle x|\, U(t) |\psi\rangle \\ &= \sum_{k=0}^{N-1} \tilde{\psi}(k)\, e^{-iE_k t} \langle x|k\rangle \\ &= \sum_{k=0}^{N-1} \tilde{\psi}(k)\, e^{-iE_k t} \tfrac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} kx}\end{aligned}$$

If at time t we measure the position of the particle, we will measure x_0 with probability $\dfrac{|\psi_t(x_0)|^2}{\langle\psi|\psi\rangle} = |\psi_t(x_0)|^2$.

$\Rightarrow$ Even though the particle starts at $x=0$, after time passes we have a nonzero chance of measuring any position for it.

This is an example of Heisenberg's uncertainty principle, which basically asserts that the position basis and the momentum basis are completely incompatible.

$\rightsquigarrow$ essentially because $|\langle x|k\rangle| > c > 0$.

To prove the theorem on $\exp: B(V) \to B(V)$, we use the following.

Thm: If V is Banach, then $B(V)$ is Banach with the operator norm.

We can prove this when V is finite-dim since then $B(V) = \mathrm{End}(V)$ is a finite dimensional normed space.

$\Rightarrow$ The Thm follows from the last problem.

We showed the following in a warm-up

Lemma: If a series converges absolutely in a Banach space, then the series converges.

Recall that the series

$$\sum_{n=0}^{\infty} v_n$$

converges absolutely if the series

$$\sum_{n=0}^{\infty} \|v_n\|$$

converges in $\mathbb{R}$.

Now we can prove that if L is a bounded operator on a Banach space V, then $\sum_{n=0}^{\infty} \frac{L^n}{n!}$ converges.

$\Rightarrow$ NTS $\sum_{n=0}^{\infty} \frac{\|L^n\|}{n!} < \infty$.

Note that $\|Lv\| \leq \|L\| \|v\|$ (using properties of inf) so

$$\|L_1 L_2\| \leq \|L_1\| \|L_2\|$$

$$\Rightarrow \|L^n\| \leq \|L\|^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\|L^n\|}{n!} \leq \sum_{n=0}^{\infty} \frac{\|L\|^n}{n!} = e^{\|L\|} < \infty.$$

Since $B(V)$ is a Banach space, it follows that $\exp(L)$ is well-defined for every bounded operator L.

One last thing:

Thm: A linear operator L on a normed space is continuous iff it is bounded.

So one reason for liking Banach spaces is that we can define things like exponential functions that have our desired properties.

Another reason is that we will need our inner product spaces to be complete in order for results like the Spectral Theorem to hold. We need the spectral theorem to do Fourier transforms, and we need Fourier transforms to have momentum operators!

So to do quantum physics for a particle on more complicated spaces than $\mathbb{Z}/N$, like $\mathbb{Z}$, S^1, or $\mathbb{R}^n$, we need

most common!

a complete inner product space

In other words,

a Hilbert space